Chapter 19. Allowing for Stochastic Interest Rates in the Black-Scholes Model
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1. Allowing for Stochastic Interest Rates in the B-S Model
   - The Hedging Portfolio
   - Solving for the Option Price
Introduction

- Chapter 12 and Chapter 15 considered a relaxation of one of the key assumptions of the Black-Scholes framework, namely that the asset price changes follow a geometric Brownian motion.
- Another crucial assumption of the Black-Scholes framework; a constant interest rate over the life of the option.
- We consider the specific case of stock options and retain all the assumptions of the original Black-Scholes model, except that we now allow interest rates to vary stochastically.
Introduction

- The issue raised by the interest rate being stochastic is that we can no longer discount the expected future option pay-off using the deterministic discount factor $\exp[-r(T - t)]$.

- Now we must find a corresponding **stochastic discount factor** and adjust the pricing relationship accordingly.

- Our discussion follows closely that of Merton (973b); uses **the price of a bond having the same maturity as the option** to capture the effect of stochastic interest rates.
Introduction

Let $T$ denote option maturity

$$P(t, T) = \begin{cases} \text{the price at time } t \text{ of a riskless discount} \\ \text{bond which pays $1$ at time } T(> t). \end{cases}$$

We allow for stochastic interest rates by allowing the bond price to vary stochastically;

$$\frac{dP}{P} = \alpha(P, t) dt + \delta(P, t) dv,$$  \hspace{1cm} (1)
Introduction

- Here \( dv \) is the increment of a Wiener process which is the source of the uncertainty in the evolution of the bond price\(^1\), and \( \tau(= T - t) = \) time to maturity.
- The mean return \( \alpha(\tau) \) could depend on the level of bond prices as well as the time-to-maturity.

\(^1\)Since the sources of uncertainty may be different for bonds of differing maturities we should write \( dv(t, \tau) \) to denote this dependency on \( \tau \), and since the \( dv \) at differing maturities would not be perfectly correlated we would have

\[
\mathbb{E}[dv(t, \tau_1)dv(t, \tau_2)] = \rho_{12}dt. \tag{2}
\]

However since we only consider a bond having the same maturity as the option we do not need such a notation here. Furthermore we shall see more clearly how to capture the correlation between bonds at different maturities when we come to study the Heath-Jarrow-Morton model.
Introduction

- To ensure that $P(T, T) = 1$ we could use a Brownian bridge process (Section 6.3.6). In fact, the choice

$$\alpha(P, t) = \frac{1}{2} \delta^2 - \frac{\ln P(t, T)}{(T - t)},$$

$$\delta(P, t) = \delta.$$

- The traditional assumption of constant interest rate is recovered by setting $\delta(P, t) = 0$ and $\alpha(P, t) = r$:

$$P(t, T) = e^{-r(T-t)}. \quad (3)$$
Introduction

- We retain the assumption that the stock price follows

\[
\frac{dS}{S} = \mu dt + \sigma dz. \tag{4}
\]

- We allow for correlation between the Wiener increments \(dv\) and \(dz\), i.e.,

\[
\mathbb{E}[dvdz] = \rho dt. \tag{5}
\]
Introduction

Figure 1: Illustrating stock prices and bond prices (of same maturity as the option) in the B-S world with a stochastic interest rate
The Hedging Portfolio

- The option value will be a function of both $S$ and $P$ as well as the time, i.e.,

$$f = f(S, P, t) \quad (6)$$

- The pricing of the option here falls into the case of pricing a derivative security dependent on several underlying state variables (Chapter 10).

- Ito’s lemma ⇒

$$\frac{df}{f} = \mu_f dt + \sigma_{fz} dz + \sigma_{fv} dv, \quad (7)$$
The Hedging Portfolio

where

\[ \mu_f = \frac{1}{f} \left[ \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \alpha P \frac{\partial f}{\partial P} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho \sigma \delta S \frac{\partial^2 f}{\partial S \partial P} + \frac{1}{2} \delta^2 P^2 \frac{\partial^2 f}{\partial P^2} \right] \]

\[ \sigma_{f_z} = \frac{\sigma S}{f} \frac{\partial f}{\partial S}, \quad \text{and} \quad \sigma_{f_P} = \frac{\delta P}{f} \frac{\partial f}{\partial P}. \]  

(8)

- We form a portfolio of **stock**, the **option** and **riskless bonds** having time to maturity \( \tau \) equal to the expiration of the option.
- Let \( Q_1, Q_2 \) and \( Q_3 \) denote respectively the number of **dollars** of the portfolio invested in the stock, the option and the bonds;

\[ Q_1 + Q_2 + Q_3 = 0. \]  

(9)
The Hedging Portfolio

- The instantaneous dollar return on the portfolio is given by

\[
Q_1 \frac{dS}{S} + Q_2 \frac{df}{f} + Q_3 \frac{dP}{P}
\]

\[
= [Q_1(\mu - \alpha) + Q_2(\mu_f - \alpha)]dt
+ [Q_1\sigma + Q_2\sigma_{f_z}]dz + [Q_2\sigma_{f_v} - (Q_1 + Q_2)\delta]dv.  \tag{11}
\]

- Choose the proportions \(Q_1, Q_2\) so that the stochastic \(dz\) and \(dv\) terms vanish i.e.

\[
Q_1\sigma + Q_2\sigma_{f_z} = 0, \tag{12}
\]

\[
Q_2\sigma_{f_v} - (Q_1 + Q_2)\delta = 0. \tag{13}
\]
The Hedging Portfolio

- From the 1st eqn.

\[
\frac{Q_1}{Q_2} = -\frac{\sigma_f}{\sigma}
\]

and from the 2nd eqn.

\[
\frac{Q_1}{Q_2} = \frac{\sigma_f}{\delta} - 1.
\]

- From these two eqns. we find that

\[
\frac{\sigma_f}{\sigma} = 1 - \frac{\sigma_f}{\delta},
\]

\sim \text{ which becomes (from the definitions (8) of } \sigma_f \text{ and } \sigma_f \text{)}

\[
f = S \frac{\partial f}{\partial S} + P \frac{\partial f}{\partial P}. \tag{14}
\]
The Hedging Portfolio

With this choice of $Q_1$, $Q_2$ the hedging portfolio is riskless and its instantaneous dollar return is given by

$$
\left[ Q_1(\mu - \alpha) - \frac{Q_1\sigma}{\sigma_{fz}}(\mu_f - \alpha) \right] dt
$$

$$
= Q_1\sigma \left[ \frac{\mu - \alpha}{\sigma} - \frac{\mu_f - \alpha}{\sigma_{fz}} \right] dt.
$$

This instantaneous dollar return should be zero, as involves zero net investment. Thus

$$
\frac{\mu - \alpha}{\sigma} = \frac{\mu_f - \alpha}{\sigma_{fz}}. \quad (15)
$$
The Hedging Portfolio

- A modified form of equality of risk adjusted excess return of risky assets in the portfolio.
  
  Here, the constant risk free rate \( r \) is replaced by \( \alpha \), the instantaneous return on a riskless bond having the same maturity as the option.

- Applying the definitions (8) of \( \mu_f \) and \( \sigma_{f^z} \) we may reduce (15) to the partial differential equation

\[
\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \alpha P \frac{\partial f}{\partial P} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho \sigma \delta S P \frac{\partial^2 f}{\partial S \partial P} + \frac{1}{2} \delta^2 P^2 \frac{\partial^2 f}{\partial P^2} - \alpha f = S \frac{\partial f}{\partial S} (\mu - \alpha),
\]
The Hedging Portfolio

- which simplifies to

\[
\frac{\partial f}{\partial t} + \alpha \left( S \frac{\partial f}{\partial S} + P \frac{\partial f}{\partial P} \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = f \text{ by (14)}
\]

\[
+ \rho \sigma \delta S P \frac{\partial^2 f}{\partial S \partial P} + \frac{1}{2} \delta^2 P^2 \frac{\partial^2 f}{\partial P^2} - \alpha f = 0.
\]

- So (15) has reduced to the p.d.e.

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho \sigma \delta S P \frac{\partial^2 f}{\partial S \partial P} + \frac{1}{2} \delta^2 P^2 \frac{\partial^2 f}{\partial P^2} + \frac{\partial f}{\partial t} = 0. \quad (16)
\]
The Hedging Portfolio

- Equation (16) needs to be solved subject to the boundary conditions appropriate for the option of interest, for instance a European call or put option.

- For future reference note that it is often convenient to consider (16) in terms of time to maturity $\tau = T - t$, so that

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho \sigma \delta S P \frac{\partial^2 f}{\partial S \partial P} + \frac{1}{2} \delta^2 P^2 \frac{\partial^2 f}{\partial P^2} = \frac{\partial f}{\partial \tau} \quad (17)$$

- To obtain the last equation note that

$$\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial \tau} = - \frac{\partial f}{\partial t}$$
Solving for the Option Price

- The relationship (14) suggests that \( f \) is first degree homogeneous in \((S, P)\).

- The function \( g(u, v) \) is homogeneous of degree \( n \) in \((u, v)\) if \( g(\lambda u, \lambda v) = \lambda^n g(u, v) \). Differentiation with respect to \( \lambda \) yields

\[
ug_u(\lambda u, \lambda v) + vg_v(\lambda u, \lambda v) = n\lambda^{n-1} g(u, v).
\]

Setting \( n = 1 \), then \( \lambda = 1 \) yields an expression of the form (14).

- Then the type of transformation used in Appendix 9.1 to solve the basic Black-Scholes eqn. can be used to reduce the dimensionality of the pricing eqn. (16).
Solving for the Option Price

- Introduce the new state variable \( X \) defined by

\[
X = \frac{S}{EP}.
\]  

(18)

- Here \( P \) is used to discount between maturity and current time, so that \( X \) may be interpreted as the price per share of stock in units of the present value of the exercise price.

\[ \sim \] In the case of a constant interest rate, \( P(t, T) = e^{-r(T-t)} \), and we recover the change of variable given in (9.39).

- Before proceeding, note that since \( S \) and \( P \) are driven by the s.d.e.s (1) and (4) Ito’s lemma \( \Rightarrow \)

\[
\frac{dX}{X} = [\mu - \alpha + \delta^2 - \rho\sigma\delta]dt + \sigma dz - \delta dv.
\]  

(19)
Solving for the Option Price

- The instantaneous variance of the return on $X$ is given by

$$\text{var} \left[ \frac{dX}{X} \right] = \mathbb{E}[(\sigma dz - \delta dv)^2]$$

$$= \mathbb{E}[\sigma^2 (dz)^2 + \delta^2 (dv)^2 - 2\sigma\delta dz dv]$$

$$= (\sigma^2 + \delta^2 - 2\rho\sigma\delta)dt$$

$$\equiv V^2(\tau)dt, \quad (20)$$

where we define

$$V^2(\tau) = \sigma^2 + \delta^2 - 2\rho\sigma\delta. \quad (21)$$

- Switch the time unit to $\tau = T - t$. 
Solving for the Option Price

- Motivated by the homogeneity property of $f$, define the price

$$ h(X, \tau) = \frac{f(S, P, \tau)}{EP(\tau)}. \quad (22) $$

- To understand the last equation rate that if $f$ is first order homogenous in $S, P$, then

$$ f(\lambda S, \lambda P, \tau) = \lambda f(S, P, \tau). $$

Choose

$$ \lambda = \frac{1}{EP} \Rightarrow f\left(\frac{S}{EP}, \frac{1}{E}, \tau\right) = \frac{f(S, P, \tau)}{PE} \Rightarrow h(X, \tau) = \frac{f(S, P, \tau)}{EP}, $$

since

$$ X = \frac{S}{EP}. $$
Solving for the Option Price

In transforming from the variables \((f, S, P, \tau)\) to \((h, X, \tau)\) we use

\[
\begin{align*}
\frac{\partial f}{\partial S} &= \frac{\partial h}{\partial X}, \\
\frac{\partial f}{\partial P} &= Eh - \frac{S}{P} \frac{\partial h}{\partial X}, \\
\frac{\partial^2 f}{\partial S^2} &= \frac{1}{EP} \frac{\partial^2 h}{\partial X^2}, \\
\frac{\partial f}{\partial \tau} &= EP \frac{\partial h}{\partial \tau}.
\end{align*}
\]
To derive the expressions on the previous slide note that:

\[
\frac{\partial f}{\partial S} = EP \frac{\partial h}{\partial X} \cdot \frac{\partial X}{\partial S} = EP \cdot \frac{1}{EP} \cdot \frac{\partial h}{\partial X} = \frac{\partial h}{\partial X},
\]

\[
\frac{\partial f}{\partial P} = Eh + EP \frac{\partial h}{\partial X} \frac{\partial X}{\partial P} = Eh + EP \left( -\frac{S}{EP^2} \right) \frac{\partial h}{\partial X} = Eh - \frac{S}{P} \frac{\partial h}{\partial X},
\]

\[
\frac{\partial^2 f}{\partial S^2} = \frac{\partial}{\partial X} \left( \frac{\partial h}{\partial X} \right) \cdot \frac{\partial X}{\partial S} = \frac{1}{EP} \frac{\partial^2 h}{\partial X^2},
\]

\[
\frac{\partial^2 f}{\partial S \partial P} = \frac{\partial}{\partial X} \left( \frac{\partial h}{\partial X} \right) \frac{\partial X}{\partial P} = - \frac{S}{EP^2} \frac{\partial^2 h}{\partial X^2},
\]

\[
\frac{\partial^2 f}{\partial P^2} = E \frac{\partial h}{\partial X} \frac{\partial X}{\partial P} + \frac{S}{P^2} \frac{\partial h}{\partial X} - \frac{S}{EP^2} \frac{\partial^2 h}{\partial X^2} \cdot \frac{\partial X}{\partial P} = \frac{S^2}{EP^3} \frac{\partial^2 h}{\partial X^2},
\]

\[
\frac{\partial f}{\partial \tau} = EP \frac{\partial h}{\partial \tau}.
\]
Solving for the Option Price

From the above we find that

\[
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho \sigma \delta S P \frac{\partial^2 f}{\partial S \partial P} + \frac{1}{2} \delta^2 P^2 \frac{\partial^2 f}{\partial P^2} = \frac{1}{2} V^2(\tau) X^2 \frac{\partial^2 h}{\partial X^2} EP,
\]

so the second-order p.d.e. (16) for \( f \) in terms of \( S, P \) and \( \tau \) has been reduced to the first-order p.d.e.

\[
\frac{1}{2} V^2(\tau) X^2 \frac{\partial^2 h}{\partial X^2} - \frac{\partial h}{\partial \tau} = 0,
\]

for \( h \) in terms of \( X \) and \( \tau \).
Solving for the Option Price

In the case of a European call option equation (23) must be solved subject to the boundary condition

\[ h(0, \tau) = 0, \]

and initial condition

\[ h(X, 0) = \max[X - 1, 0]. \]
Solving for the Option Price

- Solve eqn. (23) using the solution framework of Chapter 9.
- In eqn. (9.7), if we set \( q(t) = r(t) = 0 \), then
  \[
  \frac{1}{2} \sigma^2(t) S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} = 0.
  \]
  (24)

  This is basically eqn. (19.22) with \( \sigma(t) \rightarrow V(t), f \rightarrow h, S \rightarrow X \).

- Apply the solution (9.28) with
  \[
  E = 1, \\
  \bar{r} = 0, \\
  \bar{\sigma}^2 = \bar{V}^2 = \frac{1}{T-t} \int_t^T V^2(s) ds = \frac{1}{T-t} \int_0^{T-t} V^2(u + t) du \\
  = \frac{1}{\tau} \int_0^\tau V^2(u + t) du, \quad \text{(change of variable } u = s - t).\]
Solving for the Option Price

The solution then becomes

\[ h(X, \tau) = X \mathcal{N}(d_1) - 1 \mathcal{N}(d_2), \]

\[ \sim \quad \text{with} \]

\[ d_1 = \frac{\ln(X) + \bar{V}^2(T - t)}{\bar{V} \sqrt{T - t}}, \quad d_2 = d_1 - \bar{V} \sqrt{T - t}. \]
Solving for the Option Price

- Changing back to the original variables:

\[
\frac{f(S, P, \tau)}{EP} = \frac{S}{EP} N(d_1) - N(d_2)
\]

i.e.

\[
[8cm] \quad f(S, P, \tau) = SN(d_1) - EPN(d_2) \tag{25}
\]

with

\[
d_1 = \frac{\ln(X/1) + \frac{\bar{V}^2}{2}(T - t)}{\bar{V}\sqrt{T - t}} = \frac{\ln(S/EP) + \frac{\bar{V}^2}{2}\tau}{\bar{V}\sqrt{\tau}}
\]

\[
= \frac{\ln(S/E) - \ln P(\tau) + \frac{\tau}{2}\bar{V}^2}{\bar{V}\sqrt{\tau}}
\]

and

\[
d_2 = d_1 - \bar{V}\sqrt{\tau}.
\]
Solving for the Option Price

- The solution technique originally used by Merton involved various changes of variables, and we reproduce his analysis in Appendix 1.
- The pricing formula (25) generalises the Black-Scholes formula in a very natural way.
  - The bond with same maturity as the option is used to do stochastic discounting and the “average” volatility \( \bar{V} \) replaces the \( \sigma \) of the standard case.
- If we adopt the common practice of calculating an implied \( \bar{V} \) from market data then there is no need to estimate separately the \( \rho, \delta(\tau) \) and \( \sigma \).
  - This observation also helps to explain the robustness of the Black-Scholes model, (when used with implied volatility,)
  - as the volatility so calculated is observationally compatible with a wide class of deterministic time functions of \( \sigma, \rho \) and \( \delta \), and not just with a constant \( \sigma \).
Appendix 1. Solving the P.D.E. by Change of Variable
Solving the P.D.E. by Change of Variable

1. If we introduce the new time variable\(^2\)

\[
\theta = \int_0^\tau V^2(s)\,ds,
\]  
(26)

and define

\[
g(X, \theta) = h(X, \tau(\theta)),
\]

2. then \(g\) satisfies

\[
\frac{1}{2} X^2 \frac{\partial^2 g}{\partial X^2} - \frac{\partial g}{\partial \theta} = 0,
\]  
(27)

subject to

\[
g(0, \theta) = 0,
\]  
\[
g(X, 0) = \max[0, X - 1].\]  
(28)

\(^2\)Note that eqn. (26) defines a functional relationship between \(\tau\) and \(\theta\).
Solving the P.D.E. by Change of Variable

- We can interpret (27) as the Black-Scholes option pricing eqn. for an option with time $\theta$ to maturity, exercise price of one dollar, when the underlying stock has variance of unity and the market rate of interest is zero.

~ So we can use the known solution to write

$$g(X, \theta) = X \mathcal{N}(d_1) - \mathcal{N}(d_2),$$

(29)

where

$$d_1 = \frac{\ln X + \frac{1}{2}\theta}{\sqrt{\theta}},$$

$$d_2 = d_1 - \sqrt{\theta}.$$
Solving the P.D.E. by Change of Variable

- Working back through the transformations we obtain

\[ f(x, P, \tau) = EP(\tau)g \left( \frac{x}{EP(\tau)}, \int_0^\tau V^2(s) ds \right). \] (30)

- In addition to the usual inputs of the Black-Scholes model, (30) requires \( P(\tau) \) as well as \( \rho \) and \( \delta(\tau) \).
Solving the P.D.E. by Change of Variable

If we define

\[ \bar{V}^2 = \frac{1}{\tau} \int_0^\tau V^2(s) ds. \]  \hspace{1cm} (31)

and make use of (29) then the expression for the option price in (30) can be written

\[ f(S, P, \tau) = SN(d_1) - EP(\tau)N(d_2), \] \hspace{1cm} (32)

where

\[ d_1 = \ln\left(\frac{S}{EP(\tau)}\right) + \frac{\tau}{2} \bar{V}^2 = \ln\left(\frac{S}{E}\right) - \ln P(\tau) + \frac{\tau}{2} \bar{V}^2 \]

and

\[ d_2 = d_1 - \bar{V} \sqrt{\tau}. \]