Chapter 23. Interest Rate Derivatives - One Factor Spot Rate Models

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Introduction

- This chapter; the problem of pricing options on interest rate derivative securities.
- The essential feature of this problem is that we need to take account of the stochastic nature of interest rates.
- Chapter 19 illustrated one approach to this problem, namely modelling the price of pure discount bonds as a stochastic process and making this one of the stochastic factors upon which the value of the option depends.
- The general approach is due to Merton (973b).
Introduction

There are however a number of practical difficulties in attempting to implement this approach.

- In particular it requires specification of the average expected return variance over the time interval to maturity, together with the covariance between return and the instantaneous short term rate.
- It is not clear in practice how best to estimate these variances and covariances.

Nevertheless, Merton’s approach has guided the development of many of the subsequent interest rate option models.

A characteristic of the stock option model is that there is one basic approach to which can be added embellishments to account for different stochastic processes for the underlying asset (e.g. a jump-diffusion process) or to account for different boundary conditions (e.g. European or American options).
Introduction

- For interest rate contingent claims however there does not seem to be one basic approach but rather a range of alternative approaches.

- These differ according to what is taken as the underlying factor, which is usually one of the instantaneous spot interest rate, the bond price or the forward rate.

- Further, some models are presented in a discrete time framework and some in a continuous time framework.

- An important distinction between alternative approaches is whether the initial term structure (i.e. the currently observed yield curve) is itself to be modelled or to be taken as given.

  ∼ This modelling choice will determine whether the resulting models involve the market price of interest rate risk.
In this chapter we survey models of interest rate derivatives which take the instantaneous spot rate of interest as the underlying factor.

The by-now familiar continuous hedging argument is extended so as to model the term structure of interest rates and from this other interest rate derivative securities.

This basic approach is due to Vasicek (1977) and hence we shall often refer to it as the Vasicek approach.
Arbitrage Models of the Term Structure

- We consider the perspective of an investor standing at time 0 and observing various market rates that enable him/her to compute the initial forward $f(0, T)$ (and so the initial bond price curve $P(0, T)$) for any maturity $T$.

- This investor wishes to price at any time $t ( < T)$ a pure default-free discount bond that pays $1$ at time $T$.

- The investor seeks the arbitrage-free bond price that does not allow the possibility of riskless arbitrage opportunities between bonds of differing maturities.

- Furthermore the investor wishes the bond price so obtained to be consistent with the currently observed initial bond price curve.
Arbitrage Models of the Term Structure

Fig. 1 illustrates the time line for the bond-pricing problem.

Figure 1: The time line for the bond-pricing problem
Arbitrage Models of the Term Structure

Initially, we assume that the price of a default free bond is a function of only the current short term rate of interest and time.

Thus we write $P(r(t), t, T)$ to denote the price at time $t$ of a discount bond maturing at time $T$, having maturity value of $\$1$, when the current instantaneous spot rate of interest is $r(t)$, (which is assumed to be riskless in the sense that money invested at this rate will always be paid back) i.e.,

$$P(r(T), T, T) = 1.$$  

We assume the short term rate follows the diffusion process

$$dr = \mu_r(r, t)dt + \sigma_r(r, t)dz.$$  

(2)
Arbitrage Models of the Term Structure

- By Ito’s lemma

\[
\frac{dP}{P} = \mu_P(r, t, T)dt + \sigma_P(r, t, T)dz, \tag{3}
\]

\(~\text{where}~\)

\[
\mu_P(r, t, T) = \frac{1}{P} \left( \frac{\partial P}{\partial t} + \mu_r \frac{\partial P}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} \right), \tag{4}
\]

\[
\sigma_P(r, t, T) = \frac{\sigma_r}{P} \frac{\partial P}{\partial r}. \tag{5}
\]

- Consider an investor who at time \(t\) invests $1 in a hedge portfolio containing two default free bonds maturing at times \(T_1\) and \(T_2\) respectively; held in the dollar amounts \(Q_1\) and \(Q_2\).
Arbitrage Models of the Term Structure

- Using $P_i$ to denote the price of the bond maturing at time $T_i$, we can write

$$
\text{dollar return on the hedge portfolio } \left\{ \right. = Q_1 \frac{dP_1}{P_1} + Q_2 \frac{dP_2}{P_2} \\
= (Q_1 \mu_{P_1} + Q_2 \mu_{P_2})dt + (Q_1 \sigma_{P_1} + Q_2 \sigma_{P_2})dz,
$$

(6)

where $\mu_{P_i}, \sigma_{P_i}$ denote respectively the expected return and standard deviation of the bond of maturity $T_i$ ($i = 1, 2$).

- This return can be made certain by choosing the amounts $Q_1, Q_2$, so that

$$
\frac{Q_1}{Q_2} = -\frac{\sigma_{P_2}}{\sigma_{P_1}}.
$$

(7)
Arbitrage Models of the Term Structure

- Thus from (6) the dollar return on the now riskless hedge portfolio is

\[ (Q_1\mu_{P_1} + Q_2\mu_{P_2})dt. \]

- Absence of riskless arbitrage ⇒ this return must be the instantaneous spot rate of interest \( r \).

- Given that the original investment is $1 (i.e. \( Q_1 + Q_2 = 1 \)) then this last condition states that

\[ (Q_1\mu_{P_1} + Q_2\mu_{P_2})dt = 1 \cdot r dt. \]
Rearranging we obtain

\[ Q_1(\mu P_1 - r) + Q_2(\mu P_2 - r) = 0, \]

which when combined with (7) yields the condition for no-riskless arbitrage between bonds of any two maturities, namely

\[ \frac{\mu P_1 - r}{\sigma P_1} = \frac{\mu P_2 - r}{\sigma P_2}. \]
Arbitrage Models of the Term Structure

- Since the maturity dates $T_1$, $T_2$ were arbitrary, the ratio

$$\frac{\mu_P(r, t, T) - r(t)}{\sigma_P(r, t, T)}$$

must be independent of maturity $T$.

- Let $\lambda(r, t)$ denote the common value of this ratio for bonds of an arbitrary maturity $T$. Thus

$$\frac{\mu_P(r, t, T) - r(t)}{\sigma_P(r, t, T)} = \lambda(r, t).$$

- The quantity $\lambda$ can be interpreted as the market price of interest rate risk per unit of bond return volatility.
Arbitrage Models of the Term Structure

Thus eqn. (9) asserts that

\[ \text{in equilibrium bonds are priced so that instantaneous bond returns equal the instantaneous risk free rate of interest plus a risk premium equal to the market price of interest rate risk times instantaneous bond return volatility.} \]

Substitution from (4) of the expressions for \( \mu_P(t, T) \) and \( \sigma_P(t, T) \) results in the p.d.e. for the bond price,

\[
\frac{\partial P}{\partial t} + (\mu_r - \lambda \sigma_r) \frac{\partial P}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 P}{\partial r^2} - rP = 0, \tag{10}
\]

which must be solved subject to the boundary condition

\[ P(r(T), T, T) = 1. \tag{11} \]
To solve (10), either analytically or numerically, we need to specify the drift $\mu_r$ and diffusion $\sigma_r$ as well as form of the market price of risk term $\lambda(r, t)$. One common assumption is that this latter term is constant.

To formally derive this result, involves some very particular assumptions about how the capital market operates. These conditions are discussed briefly in the next subsection.

To give a proper theoretical basis to the choice of $\lambda(r, t)$ it would be necessary to construct a dynamic general equilibrium model and relate $\lambda(r, t)$ to investor preferences. This is the approach adopted by Cox, Ingersoll, and Ross (1985b).
The Martingale Representation

- Just as in the case of the stock option model we are able to obtain a martingale representation of the pricing relationship;
- We note from the no riskless arbitrage condition (9) that

\[ \mu_P(r, t) = r + \lambda \sigma_P(r, t) \]  \hspace{1cm} (12)

- Substitution of (12) into (3) as well as the expression for \( \sigma_P \) from (4) into (3) yields the stochastic bond price dynamics under the condition of no-riskless arbitrage viz.

\[ \frac{dP}{P} = (r + \lambda \sigma_P(r, t, T)) \, dt + \sigma_P(r, t, T) \, dz. \]  \hspace{1cm} (13)
The Martingale Representation

- Following the reasoning used in Chapter 10, define a modified Wiener process $\tilde{z}(t)$ by

$$\tilde{z}(t) = z(t) + \int_0^t \lambda(s)ds. \quad (14)$$

- Under the historical measure $\mathbb{P}$, $\tilde{z}(t)$ is not a standard Wiener process (i.e. $\mathbb{E}(\tilde{z}(t) \neq 0$ where $\mathbb{E}$ is the expectation operation under $\mathbb{P}$) but by an application of Girsanov’s theorem we can obtain an equivalent measure $\tilde{\mathbb{P}}$ under which $\tilde{z}(t)$ is a standard Wiener process (i.e. $\tilde{\mathbb{E}}(\tilde{z}(t)) = 0$ where $\tilde{\mathbb{E}}$ is the expectation operation under $\tilde{\mathbb{P}}$).

- Thus in terms of $\tilde{z}(t)$ the s.d.e. (13) for $P$ under the measure $\tilde{\mathbb{P}}$ becomes

$$\frac{dP}{P} = rdt + \sigma_P(r, t, T)d\tilde{z}. \quad (15)$$
However unlike in the stock option situation, the spot rate $r$ is here stochastic, so we need to define the money market account as

$$A(t) = e^{\int_0^t r(s)ds}.$$  \hspace{1cm} (16)

It is a simple matter to demonstrate that

$$dA = rAdt.$$  \hspace{1cm} (17)
The Martingale Representation

We then define the bond price in units of the money market account,\(^1\)

\[
Z(r, t, T) = \frac{P(r, t, T)}{A(t)},
\]

(18)

\[\sim \text{By Ito's lemma; } \frac{dZ}{Z} = \sigma_P(r, t, T)d\tilde{z}. \]

(19)

\[^{1}\text{Recall that by the rules of stochastic calculus } \]

\[
\frac{dZ}{Z} = \frac{dP}{P} - \frac{dA}{A} - \frac{dP}{P} \cdot \frac{dA}{A} + \left(\frac{dA}{A}\right)^2
\]
The Martingale Representation

Thus eqn. (19) implies \( Z(r, t, T) \) is a martingale under \( \tilde{\mathbb{P}} \), i.e.

\[
Z(r, t, T) = \tilde{\mathbb{E}}_t[Z(r(T), T, T)], \tag{20}
\]

which in terms of the original bond price can be expressed as

\[
P(r, t, T) = \tilde{\mathbb{E}}_t \left[ \frac{A(t)}{A(T)} P(r(T), T, T) \right],
\]

or since \( P(r(T), T, T) = 1 \), more simply as

\[
P(r, t, T) = \tilde{\mathbb{E}}_t \left[ e^{-\int_t^T r(s)ds} \right]. \tag{21}
\]
The Martingale Representation

- To derive the interest rate dynamics under $\bar{\mathbb{P}}$ use (14) to replace $dz$ by $(d\bar{z} - \lambda(t)dt)$ in eqn. (2);

$$dr = (\mu_r - \lambda \sigma_r)dt + \sigma_r d\bar{z}.$$ (22)

- An application of the Feynman-Kac formula\(^2\) (in particular Proposition 8.2) to (21) and (22) would take us back to the p.d.e. (10).

\(^2\)Make the identifications $x \rightarrow r$, $v(t, r) \rightarrow P(r, t, T)$, $\lambda = -1$, $f[s, x(s)] \rightarrow r(s)$.
Thus, just as in the stock option situation, we have **two representations of the bond price**, 

\[ \text{the p.d.e. (10)} \]

\[ \text{and the expectation operator (21) under the interest rate dynamics (22)}. \]

**To use these representations we need to specify the function** $\mu_r, \sigma_r$ **and also the functional form for the market price of interest rate risk** $\lambda(r, t)$. This we do in the following subsection for specific term structure models.
Before leaving this subsection we wish to emphasize the discounted cash flow interpretation of the representation (21).

The factor \( \exp(-\int_t^T r(s)ds) \) discounts back to \( t \) the dollar received at \( T \), for one particular path followed by \( r(s) \).

\( \sim \) Since \( r(s) \) is stochastic this quantity is in fact a stochastic discount factor.

\( \sim \) To obtain the discounted value at \( t \) of the $1 received at \( T \) we need to average over the range of possible paths followed by \( r(s) \) under the measure \( \tilde{\mathbb{P}} \).

\( \sim \) This is effectively what the \( \mathbb{E}_t \) does; Fig.2 illustrates this idea.
The Martingale Representation

- It is also of interest to contrast the bond price expression (21) with the corresponding expression in eqn (22.11) for a world of certainty, and we see how this is generalised in a natural way to the world of uncertainty.

- We thus have a complete analogy with the stock option price derivation of Chapter 6 and Chapter 7 with the exception that the pricing relationships here involve the market price of interest rate risk $\lambda$.

- But from our discussion in Chapter 10 this is to be expected since the underlying factor, the spot interest rate $r$, is not a traded factor.
The Martingale Representation

Figure 2: Typical paths for the $r$ process over $[t, T]$. Equation (21) averages the quantity $e^{-\int_t^T r(s)ds}$ over many such paths under the $\tilde{\mathbb{P}}$ measure.
Some Specific Term Structure Models

- A variety of term structure models are obtained
  - by specifying different forms for $\mu_r(r, t)$ and $\sigma_r(r, t)$ in the interest rate process, eqn. (2),
  - and/or different forms for the market price of risk term.
The Vasicek Model

- The Vasicek Vasicek (1977) model holds a special place in the interest rate term structure literature as it was the earliest model. Its basic assumptions are to take

\[ \mu_r(r, t) = \kappa(\gamma - r) \text{ and } \sigma_r(r, t) = \sigma, \]  

(23)

with \( \kappa > 0 \) and \( \sigma > 0 \) are constant, and also assume a constant market price of interest rate risk \( \lambda \).

- Setting \( \theta = \kappa\gamma - \lambda\sigma \), the bond pricing p.d.e. (10) becomes

\[ \frac{\partial P}{\partial t} + (\theta - \kappa r) \frac{\partial P}{\partial r} + \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial r^2} - rP = 0. \]  

(24)
The Vasicek Model

- To solve this p.d.e. we seek functions \( a(t, T) \) and \( b(t, T) \) such that \(^3\)

\[
P(t, T) = e^{-a(t,T)-b(t,T)r(t)}. \tag{25}
\]

- In order that the boundary condition (11) be satisfied for all possible \( r(t) \) it must be the case that \( a(t, T) \) and \( b(t, T) \) satisfy

\[
a(T, T) = 0 \text{ and } b(T, T) = 0. \tag{26}
\]

\(^3\text{One motivation for the functional form (25) is that when } r \text{ is constant, we have } P(t, T) = e^{-r(T-t)}, \text{ and (25) is an obvious generalisation of this relation.}\)
The Vasicek Model

- From (25) we note that

\[
\frac{\partial P}{\partial r} = -bP, \quad \frac{\partial^2 P}{\partial r^2} = b^2 P \text{ and } \frac{\partial P}{\partial t} = (-a_t - b_t r)P. \quad (27)
\]

- Substituting these relations into (24) and gathering terms in powers of \(r(r^0 = 1)\) we obtain

\[
\left[-a_t - b\theta + \frac{1}{2}\sigma^2 b^2\right] + [-b_t + \kappa b - 1]r = 0. \quad (28)
\]

- If (28) is to hold for all \(t\) and all \(r\) it must be the case that each bracket is separately equal to zero.

---

\(^4\)We employ the notation \(a_t = \frac{\partial}{\partial t} a(t,T), \ b_t = \frac{\partial}{\partial t} b(t,T).\)
The Vasicek Model

Thus we obtain for \( a \) and \( b \) the ordinary differential eqns.

\[
b_t = \kappa b - 1, \tag{29}\]

\[
a_t = -b\theta + \frac{1}{2}\sigma^2 b^2, \tag{30}\]

which must be solved subject to the boundary conditions (26).
The Vasicek Model

From (29) we obtain\(^5\)

\[
b(t, T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}.
\] (31)

Substituting (31) into (30), integrating from \(t\) to \(T\) and, using the boundary condition \(a(T, T) = 0\) we find that

\[
a(t, T) = \int_t^T \left[ \theta b(s, T) - \frac{1}{2} \sigma^2 b(s, T)^2 \right] ds,
\] (32)

\(^5\text{Eqn. (29) can be re-arranged into}\)

\[
\frac{d}{dt} \left( b(t, T)e^{-\kappa t} \right) = -e^{-\kappa t}
\]

and integrating \(t\) to \(T\) we obtain

\[
b(T, T)e^{-\kappa T} - b(t, T)e^{-\kappa t} = -\int_t^T e^{-\kappa s} ds = -(e^{-\kappa t} - e^{-\kappa T})/\kappa.
\]

Use of the boundary condition \(b(T, T) = 0\) and some re-arrangement yields (31).
The Vasicek Model

which reduces to

\[ \int_t^T b(s, T) ds = \int_t^T \frac{1 - e^{-\kappa(T-s)}}{\kappa} ds \]

\[ = \int_0^{T-t} \frac{1 - e^{-\kappa u}}{\kappa} du = \frac{(T-t)}{\kappa} + \frac{1}{\kappa^2}(e^{-\kappa(T-t)} - 1) \]

and

\[ \int_t^T b^2(s, T) ds = \int_0^{T-t} \frac{(1 - e^{-\kappa u})^2}{\kappa^2} du \]

\[ = \frac{1}{\kappa^2} \left[ (T-t) + \frac{2}{\kappa}(e^{-\kappa(T-t)} - 1) - \frac{1}{2\kappa}(e^{-2\kappa(T-t)} - 1) \right] \]

\[ = \frac{(T-t)}{\kappa^2} - \frac{1}{2\kappa^3} \left\{ (e^{-\kappa(T-t)} - 1)^2 - 2(e^{-\kappa(T-t)} - 1) \right\} . \]
The Vasicek Model

The Vasicek Model

\[ a(t, T) = \left( \frac{\theta}{\kappa} - \frac{\sigma^2}{2\kappa^2} \right) (T - t) + \left( \frac{\theta}{\kappa^2} - \frac{\sigma^2}{2\kappa^3} \right) (e^{-\kappa(T-t)} - 1) \]

\[ + \frac{\sigma^2}{4\kappa^3} (e^{-\kappa(T-t)} - 1)^2. \]  

(33)

The corresponding expression for the yield to maturity is

\[ \rho(t, T) = - \ln P(t, T)/(T - t) = (a(t, T) + b(t, T)r(t))/(T - t) \]

\[ = \left( \frac{\theta}{\kappa} - \frac{\sigma^2}{2\kappa^2} \right) \left( 1 + \frac{e^{-\kappa(T-t)} - 1}{\kappa(T - t)} \right) \]

\[ + \frac{\sigma^2}{4\kappa^3} \frac{(e^{-\kappa(T-t)} - 1)^2}{T - t} + \frac{1 - e^{-\kappa(T-t)}}{T - t} r(t). \]  

(34)
The Vasicek Model

- By letting \((T - t) \to \infty\) we find that the yield at infinite maturity is given by \(^7\)

\[
\rho_\infty = \frac{\theta}{\kappa} - \frac{\sigma^2}{2\kappa^2}.
\]  

(35)

- One may then express the bond price as

\[
P(r, t, T) = \exp \left[ \frac{(e^{-\kappa(T-t)} - 1)}{\kappa} (r - \rho_\infty) - \rho_\infty (T - t) - \frac{\sigma^2}{4\kappa^3} (e^{-\kappa(T-t)} - 1)^2 \right].
\]  

(36)

\(^7\)One could use this insight to infer a value for the unknown factor \(\theta\) from the currently observed yield curve, from which one could obtain an estimate of \(\rho_\infty\).
The Vasicek Model

- **The Vasicek model is now of historical interest**, but it contains all the basic ingredients needed to deal with the more sophisticated models. Namely
  
  ~ a technique to solve the pricing p.d.e.
  ~ and the idea of relating the parameters of the model to information that can be obtained from the currently observed yield curve.

- The last observation also makes evident one of the shortcomings of the Vasicek model. By setting $t = 0$ in (36) we obtain

\[
P(r_0, 0, T) = \exp \left[ \frac{(e^{-\kappa T} - 1)}{\kappa} (r_0 - \rho_\infty) - \rho_\infty T - \frac{\sigma^2}{4\kappa^3} (e^{-\kappa T} - 1)^2 \right].
\]

(37)
The Vasicek Model

- If $\theta$ is chosen so as to match the long term yield $\rho_\infty$ then we only have two parameters, $\kappa$ and $\sigma$, left to make expression (36) consistent with the entire currently observed yield curve $P(r_0, 0, T)$.

  - Clearly this is impossible as at the most we could choose $\kappa$ and $\sigma$ to fit two points exactly, or alternatively choose them to obtain some sort of least squares fit.

- These observations suggest that one possibility to develop a model that fits the currently observed yield curve is to make at least one, if not more, of the quantities $\kappa$, $\gamma$ and $\sigma$ time varying.

  - We would then have at our disposition a whole set of values of say $\kappa$ (if it were allowed to be time varying) with which to match the theoretical model to the currently observed yield curve.
The Vasicek Model

- This is the essential insight of the Hull-White model to which we turn in the next subsection.

- However before turning to the Hull-White model we consider what the solution (25) implies for the bond price dynamics.

- We know that under $\mathbb{P}$ the bond price dynamics are given by (13). However at that point in our development we did not have an explicit expression for $\frac{\partial P}{\partial r}$.

  - The solution (25) now enables us to calculate this expression, in fact it is given in (27).
  - Substituting this expression into (13)

$$\frac{dP}{P} = (r - \lambda \sigma b(t, T))dt - \sigma b(t, T)dz.$$  \hspace{1cm} (38)
The Vasicek Model

- The last eqn. indicates that the standard deviation of bond return is $-\sigma b$, the minus sign simply indicates that a positive shock to the interest rate dynamics (i.e. a positive $dz$) results in a negative shock to the bond price dynamics. **This is merely a reflection of the fact that interest rates and bond prices are inversely related.**

- As we have seen in the discussion of the general case in Section 23.2, eqn. (38) can be transformed, under the equivalent measure $\tilde{ℙ}$, to

\[
\frac{dP}{P} = r dt - \sigma b(t, T) d\tilde{z}.
\]  

\[ (39) \]

$\sim$ The last eqn. leads, as we saw in Section 23.2 to the martingale representation (21).
The Vasicek Model

Furthermore the interest rate dynamics under $\tilde{P}$ becomes

$$dr = (\theta - \kappa r)dt + \sigma d\tilde{z}, \quad (40)$$

where we recall that $\theta$ has already been defined as $\theta = \kappa \gamma - \lambda \sigma$.

These are the dynamics with respect to which the expectation $\tilde{E}_t$ in (21) is to be calculated.
The Hull-White Model

- Hull-White (1990) take as the process for the short rate
  \[ dr = \kappa(t)(\gamma(t) - r)dt + \sigma(t)dz. \]  
  \hspace{1cm} \text{(41)}

- The difference from the Vasicek model being the time dependence of the coefficients \( \kappa(t), \gamma(t) \) and \( \sigma(t) \), the motivation being the points discussed at the conclusion of the previous subsection.

- The bond-pricing p.d.e. (10) now becomes
  \[ \frac{\partial P}{\partial t} + (\theta(t) - \kappa(t)r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 P}{\partial r^2} - rP = 0, \]  
  \hspace{1cm} \text{(42)}

\sim \text{ where we set} \hspace{1cm} \theta(t) = \kappa(t)\gamma(t) - \lambda\sigma(t). \hspace{1cm} \text{(43)}
The Hull-White Model

- The only difference from the p.d.e. (24) being the time dependence of the coefficients $\theta(t), \kappa(t)$ and $\sigma(t)$.

- It seems not unreasonable to attempt again a solution of the form (25). In fact precisely the same manipulations yield for the time coefficients $a(t, T)$ and $b(t, T)$ the ordinary differential eqns.

$$b_t = \kappa(t)b - 1,$$  \hspace{1cm} (44)

and

$$a_t = -b\theta(t) + \frac{1}{2}\sigma(t)^2b^2,$$  \hspace{1cm} (45)

- the only difference being that the two ordinary differential eqns. we must solve now have time varying coefficients.
The Hull-White Model

- If we define

\[ \mathcal{K}(t) = \int_0^t \kappa(s) \, ds, \]  

(46)

then the solution to (44) can be written

\[ b(t, T) = \int_t^T e^{\mathcal{K}(t)-\mathcal{K}(s)} \, ds. \]  

(47)

\[ ^8 \text{Note that } \frac{d}{dt} \mathcal{K}(t) = \kappa(t) \text{ so that } \frac{d}{dt} e^{\mathcal{K}(t)} = e^{\mathcal{K}(t)} \kappa(t). \text{ Multiplying across (44) by } e^{-\mathcal{K}(t)} \text{ and re-arranging we obtain} \]

\[ \frac{d}{dt} (e^{-\mathcal{K}(t)} b(t, T)) = -e^{-\mathcal{K}(t)}. \]

The result then follows by integrating \( t \) to \( T \) and following manipulations similar to those in footnote 5.
The Hull-White Model

Substituting (47) into (45) and integrating \( t \) to \( T \) yields

\[
a(t, T) = \int_t^T b(s, T)\theta(s)ds - \frac{1}{2} \int_t^T \sigma^2(s)b(s, T)^2ds. \tag{48}
\]

For general forms of the functions \( \kappa(t) \), \( \sigma(t) \) and \( \theta(t) \) it may be necessary to perform numerically the integrations in (47) and (48).

In fact to perform the integrations we would need to also have some functional form for \( \lambda \), and this would be difficult to obtain.

It turns out that we can instead find from market data the function \( \theta(t) \) (which contains \( \lambda \)) and this is (together with \( \kappa(t) \) and \( \sigma(t) \)) all we need to use the bond pricing formula.
The Hull-White Model

- Consider the bond price dynamics implied by the bond pricing formula (13) with $a(t, T)$ and $b(t, T)$ now given by (47) and (48).
- We follow exactly the corresponding manipulations for the Vasicek model that led to eqn. (38), which are not altered by the fact that $\kappa$, $\gamma$ and $\sigma$ are now time varying.
- From the general expression (25) for the bond price and equation (4)
  \[ \sigma_P(t, T) = -\sigma(t)b(t, T). \] (49)
- Thus we obtain
  \[ \frac{dP}{P} = (r - \lambda\sigma(t)b(t, T'))dt - \sigma(t)b(t, T')dz, \] (50)
- Note the time dependence of $\sigma(t)$ and the fact that $b(t, T)$ is given by eqn. (47).
The Hull-White Model

- Under the equivalent measure $\tilde{\mathbb{P}}$ the bond price dynamics are

$$\frac{dP}{P} = rdt - \sigma(t)b(t, T)d\tilde{z},$$

(51)

which leads to the martingale representation (21) as we have shown in the general case in Section ??.

- The interest rate dynamics under which $\tilde{\mathbb{E}}_t$ is calculated are given by (after setting $dz = d\tilde{z} - \lambda(t)dt$ in (41))

$$dr = (\theta(t) - \kappa(t)r)dt + \sigma(t)d\tilde{z}.$$  

(52)
The Cox-Ingersoll-Ross (CIR) Model

- Cox, Ingersoll, and Ross (1985a) (CIR) consider the interest rate process

\[ dr = \kappa(t)(\gamma - r)dt + \sigma \sqrt{r} dz. \]  

(53)

- As we discussed in Section 22.3 this process guarantees non-negative (or positive if \( \kappa(t)\gamma > \sigma^2/2 \)) spot interest rate sample paths.
- Using a dynamic general equilibrium framework.
- In fact CIR employ a dynamic general equilibrium framework to derive the bond pricing equation and under specific assumptions about investor preferences, end up with a market price of risk given by (54).
The Cox-Ingersoll-Ross (CIR) Model

- In order to obtain a tractable bond pricing equation we assume that the market price of interest rate risk is a function of $r$ given by

$$\lambda(r) = \lambda \sqrt{r}, \quad (54)$$

where $\lambda$ is a constant.

- The pricing p.d.e. (10) becomes

$$\frac{1}{2} \sigma^2 r \frac{\partial^2 P}{\partial r^2} + (\kappa(t) \gamma - (\kappa(t) + \lambda \sigma) r) \frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - r P = 0. \quad (55)$$

- Given the very similar structure to the p.d.e. encountered in the Vasicek and Hull/White models (the only difference is the $r$ in front of the second derivative) we again try a solution of the same form viz.

$$P(t, T) = e^{-r b(t, T) - a(t, T)}. \quad (56)$$
The Cox-Ingersoll-Ross (CIR) Model

- Again, the condition \( P(T, T) = 1 \) can only be guaranteed if

\[
b(T, T) = 0, \quad a(T, T) = 0. \tag{57}
\]

- We note also that here

\[
\frac{\partial P}{\partial r} = -bP, \quad \frac{\partial^2 P}{\partial r^2} = b^2 P, \quad \frac{\partial P}{\partial t} = (-rb_t - a_t)P,
\]

which upon substitution into (55) and re-arrangement of terms yields

\[
[-\kappa(t)\gamma b - a_t] + r \left[ \frac{1}{2} \sigma^2 b^2 + (\kappa(t) + \lambda \sigma)b - b_t - 1 \right] = 0. \tag{58}
\]
The Cox-Ingersoll-Ross (CIR) Model

In order that this relation hold for all \( r \) and all \( t \) it must be the case that

\[
\frac{1}{2}\sigma^2 b^2 + (\kappa(t) + \lambda)b - b_t - 1 = 0, \quad (59)
\]

\[
\sim \quad \text{and}
\]

\[
-\kappa(t)\gamma b - a_t = 0. \quad (60)
\]
The Cox-Ingersoll-Ross (CIR) Model

- The difference compared to the solution of the Hull-White model is the $b^2$ term in the ordinary differential equation (59).
- This is in fact the well-known Ricatti ordinary differential equation whose solution is known. We show in Appendix ?? that the solution to (59) is

$$b(t, T) = \frac{2}{\sigma^2} \frac{[1 - e^{-\beta(T-t)}]}{[\phi_1 e^{-\beta(T-t)} - \phi_2]},$$

(61)

where

$$\phi_1 = -\frac{(\kappa(t) + \lambda)}{\sigma^2} + \frac{\beta}{\sigma^2}, \quad \phi_2 = \frac{-(\kappa(t) + \lambda)}{\sigma^2} - \frac{\beta}{\sigma^2},$$

$$\beta = \sqrt{(\kappa(t) + \lambda)^2 + 2\sigma^2}.$$

- Equation (61) appears in many different forms in the literature.
The Cox-Ingersoll-Ross (CIR) Model

- Eqn. (60) may be written

\[ \frac{da}{dt} = -\kappa(t)\gamma b(t, T), \]

which upon integration from \( t, T \) yields (using \( a(T, T') = 0 \))

\[ -a(t, T) = -\kappa(t)\gamma \int_t^T b(s, T)ds. \] (62)

- We show in Appendix ?? that (62) integrates to

\[ a(t, T) = \frac{2\kappa(t)\gamma}{\beta\sigma^2} \left[ -\beta \frac{(T-t)}{\phi_1} - \frac{(\phi_1 - \phi_2)}{\phi_1 \phi_2} \ln\left( \frac{\phi_1 - \phi_2 e^{\beta(T-t)}}{\phi_1 - \phi_2} \right) \right]. \] (63)

There are also many alternative representations of (63) in the literature.
The Cox-Ingersoll-Ross (CIR) Model

- We saw in the discussion on the Hull-White model, that in order to be able to calibrate the model to market data we needed the additional flexibility required by allowing the coefficients in (53) to be time varying.

- We can adopt exactly the same procedure with the CIR model so that any or all of the coefficients $\sigma, \kappa(t), \gamma$ and $\lambda$ in the partial differential equation (55) become time varying.

- Again we try a solution of the form (56) and eqns. (59) and (60) still emerge as the eqns. determining the coefficients $b$ and $a$. Only now it needs to be borne in mind that the coefficients $\sigma, \kappa(t), \lambda$ and $\gamma$ are time-varying.
The Cox-Ingersoll-Ross (CIR) Model

- We show in the appendix that the functional form (61) is still valid for $b(t, T)$ except that the constant $\beta$ is replaced by the time averaged function

$$\bar{\beta}(t, T) = \frac{1}{T - t} \int_t^T \beta(s) ds. \quad (64)$$

- The expression for $a(t, T)$ can only be left as the integral

$$a(t, T) = \int_t^T \kappa(s) \gamma(s) b(s, T) ds, \quad (65)$$

since the integration would in general be impossible analytically because $\bar{\beta}(t, T)$ could be a quite complicated time function.
The Cox-Ingersoll-Ross (CIR) Model

From (56) we can readily calculate that
\[
\frac{\partial P}{\partial r} = -b(t, T)P
\]
\sim \text{ hence eqn. (15) for the risk neutral bond price dynamics in the CIR case become}
\[
\frac{dP}{P} = rdt - \sigma b(t, T) \sqrt{r} d\tilde{z}.
\] (66)

The interest rate dynamics under the equivalent measure \(\tilde{P}\) are obtained by setting \(dz = d\tilde{z} - \lambda \sqrt{r} dt\) in (53) and so are given by
\[
dr = (\theta(t) - \alpha(t)r) dt + \sigma \sqrt{r} d\tilde{z}
\]
where
\[
\theta(t) = \kappa(t) \gamma(t) \quad \text{and} \quad \alpha(t) = \kappa(t) + \lambda \sigma.
\]
Calculation of the Bond Price from the Expectation Operator

- We have seen in Section 23.4 how to obtain an explicit expression for the bond price by solving the pricing p.d.e. (10) under various assumptions about $\mu_r$ and $\sigma_r$.
- It is also of interest to see how to obtain the same result by starting from the martingale or expectation operator expression (21).
- The particular spot interest rate models with which we are working provide one of the rare instances where we can carry out analytically, both the solution of the p.d.e. and the calculation of the expectation operator.
- **The key to carrying out the expectation operation in (21) is to determine the distributional characteristics of $\int_t^T r(s)ds$ under $\tilde{\mathbb{P}}$.**
Calculation of the Bond Price from the Expectation Operator

We shall now show that in the case of the Hull-White model this quantity is normally distributed with mean and variance that we calculate below.

\[ \sim \] We know from Chapter 6 how to calculate the expectation of the exponential of a normally distributed random variable.

The appropriate interest rate dynamics are given by eqn. (52), which using the quantity \( \mathcal{K}(t) \) defined by eqn. (46), can be written

\[
d(r(t)e^{\mathcal{K}(t)}) = e^{\mathcal{K}(t)}\theta(t)dt + e^{\mathcal{K}(t)}\sigma(t)d\tilde{z}.
\]

Integrating from \( t \) to \( s(< T) \) and re-arranging we find that

\[
r(s) = r(t)e^{\mathcal{K}(t)-\mathcal{K}(s)} + \int_t^s e^{\mathcal{K}(u)-\mathcal{K}(s)}\theta(u)du + \int_t^s e^{\mathcal{K}(u)-\mathcal{K}(s)}\sigma(u)d\tilde{z}(u).
\]
Calculation of the Bond Price from the Expectation Operator

Figure 3: The Region of Integration for equation (67)
Next integrate eqn. (67) from \(t\) to \(T\) to obtain
\[
\int_t^T r(s) ds = r(t) \int_t^T e^{\mathcal{K}(t) - \mathcal{K}(s)} ds + \int_t^T \left( \int_t^s e^{\mathcal{K}(u) - \mathcal{K}(s)} \theta(u) du \right) ds \\
+ \int_t^T \left( \int_t^s e^{\mathcal{K}(u) - \mathcal{K}(s)} \sigma(u) d\tilde{z}(u) \right) ds.
\]

Interchanging the order of integration in the second integral and applying Fubini's theorem (Section 22.4 version III is being used here), the last eqn. becomes
\[
\int_t^T r(s) ds = r(t) \int_t^T e^{\mathcal{K}(t) - \mathcal{K}(s)} ds + \int_t^T \left( \int_u^T e^{\mathcal{K}(u) - \mathcal{K}(s)} ds \right) \theta(u) du \\
+ \int_t^T \left( \int_u^T e^{\mathcal{K}(u) - \mathcal{K}(s)} ds \right) \sigma(u) d\tilde{z}(u). \tag{68}
\]
Calculation of the Bond Price from the Expectation Operator

- Using the definition of $b(t, T)$ at eqn. (47), eqn. (68) becomes

$$\int_t^T r(s)ds = b(t, T)r(t) + \int_t^T b(u, T)\theta(u)du + \int_t^T b(u, T)\sigma(u)d\tilde{z}(u).$$  \hspace{1cm} (69)

- Eqn. (69) implies that $\int_t^T r(s)ds$ is normally distributed (conditional on information at time $t$) since the coefficients on the right-hand side are at most time functions or involve realised stochastic quantities (in this case $r(t)$).

  The mean, $M(t)$, and variance $V^2(t)$, are easily calculated to be

$$M(t) = b(t, T)r(t) + \int_t^T b(u, T)\theta(u)du,$$  \hspace{1cm} (70)

and

$$V^2(t) = \int_t^T b(u, T)^2\sigma^2(u)du.$$  \hspace{1cm} (71)
Calculation of the Bond Price from the Expectation Operator

From the above discussion we can assert that (under $\tilde{\mathbb{P}}$)

$$\int_t^T r(s)ds \sim N(M(t), V^2(t)),$$  \hspace{1cm} (72)

and so

$$-\int_t^T r(s)ds \sim N(-M(t), V^2(t)).$$  \hspace{1cm} (73)
Calculation of the Bond Price from the Expectation Operator

Finally using the results of (iv) in Section 6.3 we obtain the result

\[ P(r, t, T) = \tilde{E}_t[e^{-\int_t^T r(s)ds}] \]

\[ = e^{-M(t) + \frac{1}{2}V^2(t)} \]

\[ = \exp \left[ -b(t, T)r(t) - \int_t^T b(u, T)\theta(u)du \right. \]

\[ + \left. \frac{1}{2} \int_t^T b(u, T)^2\sigma^2(u)du \right] \]

\[ = \exp[-b(t, T)r(t) - a(t, T)], \quad (74) \]

by making use of the definition of \(a(t, T)\) in eqn. (48).

We see that in eqn. (74) we have recovered the bond pricing formula (25) obtained by solving the p.d.e. (42).
Pricing Bond Options

- We continue to assume that the short-term rate follows the process (2). We also assume that there are no riskless arbitrage opportunities in the bond market.
- Thus the price of the discount bond of any maturity is still given by the solution to the p.d.e. (10).
- Let $C(r, t)$ denote the price at time $t$ of a call option of maturity $T_C$ written on a bond having maturity $T$ ($> T_C$). 
Pricing Bond Options

Figure 4: Time Line for The Bond Option Problem
Pricing Bond Options

By Ito’s lemma

\[
\frac{dC}{C} = \mu_C dt + \sigma_C dz,
\]

where

\[
\mu_C = \frac{1}{C} \left( \frac{\partial C}{\partial t} + \mu_r \frac{\partial C}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 C}{\partial r^2} \right),
\]

\[
\sigma_C = \frac{\sigma_r}{C} \frac{\partial C}{\partial r}.
\]

Consider an investor who at time \(t\) invests $1 in a hedge portfolio containing the bond of maturity \(T\) held in the dollar amount \(Q_P\) and the option of maturity \(T_C\) held in the dollar amount \(Q_C\).

\[\sim\] The dollar return on this hedge portfolio over time interval \(dt\) is given by
Pricing Bond Options

dollar return on the hedge portfolio \[
\left\{ \begin{array}{l}
\frac{dP}{P} + \frac{dC}{C} \\
= (Q_P \mu_P + Q_C \mu_C) dt + (Q_P \sigma_P + Q_C \sigma_C) dz.
\end{array} \right
\]

The hedge portfolio is rendered riskless by choosing \(Q_P, Q_C\) such that

\[
\frac{Q_P}{Q_C} = -\frac{\sigma_C}{\sigma_P}. \tag{78}
\]
Pricing Bond Options

- The absence of riskless arbitrage means that the hedge portfolio can only earn the same return as the original $1 invested at the risk-free rate;

\[ (Q_P \mu_P + Q_C \mu_C)dt = 1 \cdot rdt. \]  (79)

- Recalling that \( Q_P + Q_C = 1 \), the conditions (78) and (79) imply

\[ \frac{\mu_C - r}{\sigma_C} = \frac{\mu_P - r}{\sigma_P}. \]  (80)

- But by eqn. (9) we know that in an arbitrage-free bond market \( (\mu_P - r)/\sigma_P \) is equal to the market price of interest rate risk.
Pricing Bond Options

Thus we arrive at the no-riskless arbitrage condition between the option and bond markets, viz.

\[
\frac{\mu_C(t, s) - r(t)}{\sigma_C(t, s)} = \frac{\mu_P(t, s) - r(t)}{\sigma_P(t, s)} = \lambda(r, t). \quad (81)
\]

This has the now familiar interpretation that in the absence of riskless arbitrage the excess return risk adjusted on both the bond and the option are equal. The common factor to which they are equal is the market price of risk of the spot interest rate, the underlying factor.

Eqn. (81) yields the p.d.e. (10) for the bond price \( P \), and for the option price \( C \), the p.d.e.

\[
\frac{\partial C}{\partial t} + (\mu_r - \lambda \sigma_r) \frac{\partial C}{\partial r} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 C}{\partial r^2} - rC = 0, \quad (82)
\]
Pricing Bond Options

which in the case of a European call option (with exercise price $E$) on the bond must be solved on the time interval $0 < t < T_C$ subject to the boundary conditions

$$C(r(T_C), T_C) = \max[0, P(r(T_C), T_C, T) - E],$$
$$C(\infty, t) = 0.$$ \hfill (83)

The last condition is a consequence of the result that

$$P(\infty, t, T) = 0,$$

i.e. the bond value declines to zero as the interest rate becomes large.
Pricing Bond Options

- **Note the two-pass structure of the solution process.**

  - We must first solve the partial differential equation (10) with boundary condition (1) for the bond price $P(r(s), s, T)$ on the time interval $T_C \leq s \leq T$.
  
  - The value $P(r(T_C), T_C, T)$ is used in the solution of the partial differential equation (82) (in fact the same partial differential equation) via the boundary condn. in (83); this two pass procedure illustrated in Fig.5.
Pricing Bond Options

Figure 5: The two-pass procedure for solving the bond option pricing problem
Pricing Bond Options

In order to obtain the martingale representation for the option price we follow almost identical steps to those we followed in Section 23.3 to obtain the martingale representation for the bond price.

First we observe from the no-arbitrage condition (81) that

\[ \mu_C = r + \lambda \sigma_C. \]  (84)
Pricing Bond Options

- Substituting (84) into (75) the arbitrage free option price dynamics are given by

\[
\frac{dC}{C} = (r + \lambda \sigma_C)dt + \sigma_C dz.
\]

- The last eqn. may in turn be written in terms of \( \tilde{z}(t) \) (see eqn. (14)) as

\[
\frac{dC}{C} = rdt + \sigma_C d\tilde{z}(t),
\]

where we recall that under the equivalent measure \( \tilde{\mathbb{P}} \), the quantity \( \tilde{z}(t) \) is a standard Wiener process.
Pricing Bond Options

If we set

$$Y(r, t) = \frac{C(r, t)}{A(t)},$$

the option price measured in units of the money market account, then from Ito’s lemma

$$\frac{dY}{Y} = \frac{\sigma_r}{C} \frac{\partial C}{\partial r} d\tilde{z}(t).$$

The last eqn. implies that $Y$ is a martingale under $\tilde{\mathbb{P}}$, thus

$$Y(r, t) = \tilde{\mathbb{E}}_t[Y(r(T_C), T_C)],$$

which in terms of the option price itself can be expressed as

$$C(r, t) = \tilde{\mathbb{E}}_t[e^{-\int_t^{T_C} r(s) \, ds} C(r(T_C), T_C)].$$

(86)
Pricing Bond Options

- If for example we wish to price a European call option on a bond then the maturity condition is

\[ C(r(T_C), T_C) = \max[0, P(r(T_C), T_C, T) - X]. \]

- The interest rate dynamics under \( \tilde{\mathbb{P}} \) are still given by eqn. (22), viz.

\[ dr = (\mu_r - \lambda \sigma_r)dt + \sigma_r d\tilde{Z}. \]

- Application of the Feynman-Kac formula to (86) (see Proposition 8.3 will take us back to the option pricing p.d.e. (82).

- Recalling the discussion about stochastic discounting under \( \tilde{\mathbb{P}} \) at the end of Section 23.3 we see that equation (86) has an obvious expected (under \( \tilde{\mathbb{P}} \)) discounted payoff interpretation.
Pricing Bond Options

- One of the difficulties with evaluating the expectation in (86) is that one needs the joint distribution of

\[ \exp\left(-\int_t^{T_C} r(s)ds\right) \]

and

\[ C(r(T_C), T_C). \]

- Calculation of this joint distribution may be quite difficult.
Pricing Bond Options

![Diagram showing bond prices observed at time $t$]

Figure 6: Using $P(t, T_C)$ as numeraire - the forward measure
Pricing Bond Options

- A simpler calculation may be obtained by using the so-called forward measure\(^9\), which consists in choosing as numeraire a bond of maturity \(T_C\) (see Fig.6).

- That is we consider

\[
Y(r, t, T_C, T) = \frac{C(r, t, T_C, T)}{P(r, t, T_C)},
\]

\(^9\)The measure \(\mathbb{P}^*\) that we develop below is known as the forward measure because under this measure the instantaneous forward rate equals the expected future forward rate, as we show in Section (25.5).
Pricing Bond Options

- The dynamics of $Y$ under $\tilde{\mathbb{P}}$ are given by (see Section 6.6 and recall that under $\tilde{\mathbb{P}}$ the dynamics for $P$ and $C$ are given respectively by equations (15) and (85))

$$
\frac{dY}{Y} = -\sigma_P(t, T_C)(\sigma_C - \sigma_P(t, T_C))dt + (\sigma_C - \sigma_P(t, T_C))d\tilde{z}.
$$

(88)

- Equation (88) may be rearranged to

$$
\frac{dY}{Y} = (\sigma_C - \sigma_P(t, T_C))(d\tilde{z} - \sigma_P(t, T_C)dt).
$$

- Following the discussion in Section 20.1 we can define a new process

$$
z^*(t) = \tilde{z}(t) - \int_0^t \sigma_P(u, T_C)du,
$$

(89)

and a new measure $\mathbb{P}^*$ such that $z^*(t)$ is a Wiener process under this measure.
Pricing Bond Options

Thus the dynamics for $Y$ become

$$\frac{dY}{Y} = (\sigma_C - \sigma_P(t, T_C))dz^*,$$

and it follows that $Y$ is a martingale under $\mathbb{P}^*$.

Using $\mathbb{E}_t^*$ to denote expectations under $\mathbb{P}^*$, formed at time $t$, then

$$Y(r(t), t, T_C, T) = \mathbb{E}_t^* \left[ Y(r(T_C), T_C, T_C, T) \right], \quad (90)$$

Upon use of the definition of $Y$ we obtain

$$C(r(t), t, T_C, T) = P(r(t), t, T_C) \mathbb{E}_t^* \left[ C(r(T_C), T_C, T_C, T) \right]. \quad (91)$$

The difference between the expressions (86) and (91) for the value of the bond option lies in the way the stochastic discounting is done.
Pricing Bond Options

- In (86), the stochastic discounting is done along each stochastic interest rate path from $t$ and $T_C$, and since these paths are stochastic this term must appear under the expectation operator.

- In (91) the discounting from $t$ to $T_C$ is done using the bond of maturity $T_C$, which is known to the investor at time $t$ and hence this term does not need to appear under the expectation operator.

- It sometimes turns out that the expectation operation in (91) can be calculated explicitly, as we shall see in Section 23.7 for the Hull-White and CIR models.
Pricing Bond Options

- If it is necessary to evaluate the expectation in (91) by simulation then we will need the dynamics for \( r \) under \( \mathbb{P}^\ast \).

- These are easily obtained by using (89) to replace \( d\tilde{z} \) in (22) by \( dz^\ast + \sigma_P(t, T_C)dt \) so that

\[
dr = (\mu_r + \sigma_r(\sigma_P(t, T_C) - \lambda))dt + \sigma_r dz^\ast. \quad (92)
\]
Pricing Bond Options

- It is also of interest to obtain the dynamics under $\mathbb{P}^*$ for the relative bond price

$$X(r, t, T_C, T) = \frac{P(r, t, T)}{P(r, t, T_C)}.$$  \hspace{1cm} (93)

- This follows by noting that under $\mathbb{P}^*$ we have

$$\frac{dP(t, T)}{P(t, T)} = \left( r + \sigma_P(t, T) \sigma_P(t, T_C) \right) dt + \sigma_P(t, T)dz^*.$$ 

- Using the results of Section (6.6), we obtain

$$\frac{dX}{X} = \left( \sigma_P(t, T) - \sigma_P(t, T_C) \right) dz^*.$$  \hspace{1cm} (94)
Solving the Option Pricing Equation

- In this section we apply the general spot interest rate pricing framework of Section 23.6 to two special models that yield closed form solutions.

- First the Hull-White model, which assumes a Gaussian process for the spot interest rate.

- Second the CIR model which assumes a Feller or square root process for the spot interest rate.

- In both cases it is convenient to use the bond of option maturity as numeraire.

- The option pricing formula is basically Black-Scholes in the Hull-White case or Black-Scholes like in the CIR case.
The Hull White Model

- Recall eqn. (52) that for the Hull-White model the spot interest rate dynamics under $\tilde{\mathbb{P}}$ are given by

$$dr = (\theta(t) - \kappa(t)r)dt + \sigma(t)d\tilde{z},$$

where $\theta(t)$ is defined at eqn. (43).

- In this case eqn. (82) becomes

$$\frac{\partial C}{\partial t} + (\theta(t) - \kappa(t)r)\frac{\partial C}{\partial r} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 C}{\partial r^2} - rC = 0, \quad (95)$$

subject to the boundary condition (83).
In turns out that the solution to (95) can be very elegantly obtained by an application of the change of numeraire results of Chapter 20.

Instead of using the money market account as the numeraire, it is more convenient to use the price of the pure discount bond $P(r, t, T_C)$ whose maturity date is $T_C$.

From (49) with $T = T_C$ the bond return volatility for the Hull-White model is

$$\sigma_P(t, T_C) = -\sigma(t) b(t, T_C) \quad (96)$$

with $b(t, T)$ defined by (47).
The Hull White Model

- Consider the specific case of a European call bond option for which

$$C(r(T_C), T_C, T_C, T) = \left( P(r, T_C, T) - E \right)^+.$$  

- Recalling the definition of the relative bond price, see equation (93), this payoff may be written

$$C(r(T_C), T_C, T_C, T) = \left( X(T_C, T_C, T) - E \right)^+.$$  

- Substituting (96) into (94) we find that the dynamics of $X$ become

$$\frac{dX}{X} = \sigma(t) [b(t, T_C) - b(t, T)] d\tilde{z}^*(t). \quad (97)$$  

- In terms of the relative bond price $X$ we can express (91) as
The Hull White Model

\[ \frac{C(r, t, T)}{P(r, t, T)} = \mathbb{E}_t^*[ (X(T_C, T_C, T) - E)^+ ]. \]  \hspace{1cm} (98)

- Since the expectation in (98) is with respect to outcomes for the \( X \) variable, the relevant stochastic dynamics underlying the probability distn. in the calculation of \( \mathbb{E}_t^* \) is the s.d.e. (97).

- From equation (97) \( dX/X \) is normally distributed under \( \mathbb{P}^* \), with

\[ \mathbb{E}_t^* \left[ \frac{dX}{X} \right] = 0, \]  \hspace{1cm} (99)

\[ \text{var}^* \left[ \frac{dX}{X} \right] = \sigma^2(t)[b(t, T_C) - b(t, T)]^2 dt \equiv v^2(t)dt. \]  \hspace{1cm} (100)
The Hull White Model

- The calculation of the expectation in (98) with driving dynamics (97) is simply the **Black-Scholes European call option pricing problem with** $r = 0$, exercise price $E$ and time varying variance $v^2(t)$.

- Thus

$$E_t^*[(X(T_C, T_C, T) - E)^+] = X(t, T_C, T)N(d_1^*) - EN(d_2^*),$$

(101)

$$\sim \text{ where}$$

$$d_1^* = \frac{\ln(X(t, T_C, T)/E) + \bar{v}^2(T_C - t)/2}{\bar{v}\sqrt{T_C - t}},$$

$$d_2^* = d_1^* - \bar{v}\sqrt{T_C - t},$$

$$\bar{v}^2 = \frac{1}{T_C - t} \int_t^{T_C} v^2(s)ds.$$
The Hull White Model

- Substituting (101) into (98);

\[ C(r, t, T) = P(r, t, T_C)X(t, T_C, T)N(d_1^*) - EP(r, t, T_C)N(d_2^*) \]
\[ = P(r, t, T)N(d_1^*) - EP(r, t, T_C)N(d_2^*) \]

(102)

\[ \sim \text{ with } d_1^* \text{ given by} \]
\[ d_1^* = \frac{\ln \left( \frac{P(r, t, T)}{P(r, t, T_C)E} \right) + \bar{v}^2(T_C - t)/2}{\bar{v}\sqrt{T_C - t}} \]

\[ \sim \text{ and } d_2^* \text{ by} \]
\[ d_2^* = d_1^* - \bar{v}\sqrt{T_C - t}. \]

- By the put-call parity condition, the corresponding put option price can similarly be expressed as

\[ U(r, t, T) = EP(r, t, T_C)N(-d_2^*) - P(r, t, T)N(-d_1^*). \]
The Hull White Model

- The structure of the option pricing formula (102) should be compared with eqn 19.23 in Chapter 19, the one obtained for the Black-Scholes model with stochastic interest rates.
- One sees that they are identical in structure if one replaces the underlying traded asset (the stock $S$) of Chapter 19 with the underlying traded asset (the bond $P$) of the current situation.
The CIR Model

In the case of the CIR model with the interest rate process given by (53), eqn. (82) becomes

$$\frac{\partial C}{\partial t} + [\alpha(\gamma - r) - \lambda \sigma r] \frac{\partial C}{\partial r} + \frac{1}{2} \sigma^2 r \frac{\partial^2 C}{\partial r^2} - rC = 0. \quad (103)$$
The CIR Model

Eqn. (103) can also be solved by using the change of numeraire ideas of Chapter 20.

The derivation follows exactly the same lines as in the previous subsection, the only difference is that now the bond price dynamics are given by (66).

As a result the dynamics for $X$ are given by

$$\frac{dX}{X} = v(t)\sqrt{r}dW^*, \quad (104)$$

where

$$v(t) = \sigma[b(t, T) - b(t, T_C)], \quad (105)$$

with $b(t, T)$ given by eqn. (61).
The CIR Model

- The Kolmogorov eqn. associated with eqn. (104) is

\[
\frac{1}{2} v^2(t) r \frac{\partial^2 \pi}{\partial r^2} + \frac{\partial \pi}{\partial t} = 0. \tag{106}
\]

The p.d.f. arising from eqn. (106) is essentially given by equation 22.19 in Chapter 23 in the limit \( \kappa \to 0 \).

Integration of the call option payoff with respect to this distn. yields the option price.
The CIR Model

For instance, if the boundary condition is given by (83) the expression for the option price turns out to be

\[
C(r, t, T_C; T, K) = P(r, t, T) \chi^2 \left( 2r^* [\phi + \psi + B(T_C, T)]; \frac{4\alpha \gamma}{\sigma^2}, \frac{2\phi^2 r e^{\xi(T_C - t)}}{\phi + \psi + B(T_C, T)} \right)
\]

(107)

\[
- EP(r, t, T_C) \chi^2 \left( 2r^* [\phi + \psi]; \frac{4\alpha \gamma}{\sigma^2}, \frac{2\phi^2 r e^{\xi(T_C - t)}}{\phi + \psi} \right),
\]

(108)
The CIR Model

\[ \xi \equiv \left( (\alpha + \lambda)^2 + 2\sigma^2 \right)^{1/2}, \]
\[ \phi \equiv \frac{2\xi}{\sigma^2 (e^{\xi(T_C-t)} - 1)}, \]
\[ \psi \equiv \frac{\alpha + \lambda + \xi}{\sigma^2}, \]
\[ r^* \equiv \frac{1}{B(T_C,T)} \left[ \log \left( \frac{A(T_C,T)}{E} \right) \right], \]

\( \chi^2(\cdot) \) is the noncentral chi-square distn. function and \( r^* \) is the critical interest rate below which exercise will occur, namely that obtained by solving \( E = P(r^*, T_C, T) \).
Consider again the model with interest rate dynamics under the historical measure $\mathbb{P}$ given by (41). We know the bond price is

$$P(r, t, T) = e^{-a(t, T) - b(t, T)r(t)},$$  \hspace{1cm} (109)

where

$$b(t, T) = \int_t^T e^{\mathcal{K}(t) - \mathcal{K}(s)} ds, \quad \mathcal{K}(t) = \int_0^t \kappa(s) ds,$$  \hspace{1cm} (110)

$$a(t, T) = \int_t^T b(s, T)\theta(s) ds - \frac{1}{2} \int_t^T \sigma(s)^2 b(s, T)^2 ds,$$  \hspace{1cm} (111)

and

$$\theta(t) = \kappa(t)\gamma(t) - \lambda(t)\sigma(t).$$  \hspace{1cm} (112)
Rendering Spot Rate Models

- In order to use this model we need estimates (from market data) for $\sigma(t)$, $\kappa(t)$ and $\theta(t)$.
- **Note that $\theta(t)$ impounds in itself the functions $\gamma(t)$ and $\lambda(t)$, which do not therefore need to be separately estimated**, at least for the purposes of pricing derivative securities.
- We assume that we already have estimates of $\sigma(t)$ from the prices of interest rate caps using the corresponding option pricing formula (see Section 23.7), **thus it only remains to determine $\kappa(t)$ and $\theta(t)$**.
- We assume that we also have available market information on the volatility of bonds returns of all maturities at time 0.
- We know from eqns. (13) and (109) that the volatility of bond returns is $-\sigma(t)b(t, T)$. 
Thus we assume $\sigma(0)b(0, T)$ is given as a function of maturity $T$. Putting $t = 0$ in eqn. (83) we have

$$b(0, T) = \int_0^T e^{-\mathcal{K}(s)} ds. \quad (113)$$

Differentiating with respect to maturity $T$ yields

$$\mathcal{K}(T) = - \ln \left( \frac{\partial}{\partial T} b(0, T) \right),$$

and since $\mathcal{K}'(t) = \kappa(t)$, we obtain

$$\kappa(T) = - \frac{\partial}{\partial T} \left[ \ln \left( \frac{\partial}{\partial T} b(0, T) \right) \right]. \quad (114)$$
Rendering Spot Rate Models

- Next set $t = 0$ in the bond pricing eqn. so that

$$P(r_0, 0, T) = e^{-a(0,T) - b(0,T)r_0}.$$  \hspace{1cm} (115)

- The function $P(r_0, 0, T)$ would be available from the currently observed yield curve. Consider (115) in the form

$$a(0, T) = -\ln P(r_0, 0, T) - b(0, T)r_0.$$  \hspace{1cm} (116)

- From the last eqn. $a(0, T)$ can be considered as known (from market data) as a function of $T$, we shall further assume that this function is sufficiently smooth to be at least twice differentiable.

- Thus our remaining task is to determine the function $\theta(t)$. 
Rendered Spot Rate Models

- We recall from (111) that

\[
a(0, T) = \int_0^T b(s, T)\theta(s)ds - \frac{1}{2} \int_0^T \sigma^2(s)b(s, T)^2 ds. \tag{117}
\]

- The second term on the right hand side, perhaps via numerical integration, will simply be a known function of \( T \).

  **Thus eqn. (117) constitutes an integral eqn. for the unknown function \( \theta \).**

- By a process of successive differentiations we find that

\[
\theta(T) = e^{-\kappa(T)} \frac{\partial}{\partial T} \left( e^{\kappa(T)} \frac{\partial}{\partial T} a(0, T) \right) + e^{-\kappa(T)} \frac{\partial}{\partial T} \left( e^{\kappa(T)} \frac{\partial}{\partial T} \left( \frac{1}{2} \int_0^T \sigma^2(s)b(s, T)^2 ds \right) \right). \tag{118}
\]

- Whilst equation (118) involves awkward looking algebraic expressions, its numerical evaluation would be a routine task.
Rendering Spot Rate Models

- Consider the case where $\sigma, \kappa$ are constant, so that $\theta(t)$ is the only time varying parameter.
- Now we simply have $\mathcal{K}(t) = \kappa t$, and hence

$$b(t, T) = \int_t^T e^{\kappa t - \kappa s} \, ds$$

$$= \frac{1}{\kappa} (1 - e^{\kappa (t-T)}), \quad (119)$$

from which

$$b(0, T) = \frac{1}{\kappa} (1 - e^{-\kappa T}). \quad (120)$$
Rendering Spot Rate Models

Furthermore

\[
a(t, T) = \int_t^T b(s, T)\theta(s)ds - \frac{\sigma^2}{2} \int_t^T b^2(s, T)ds, \quad (121)
\]

and so

\[
a(0, T) = \int_0^T b(s, T)\theta(s)ds - \frac{\sigma^2}{2} \int_0^T b^2(s, T)ds. \quad (122)
\]

Differentiating (122) with respect to \(T\) we obtain

\[
\frac{\partial a(0, T)}{\partial T} = \int_0^T \theta(s) \frac{\partial}{\partial T} \left( \frac{1}{\kappa} (1 - e^{-\kappa(T-s)}) \right) ds - \frac{\sigma^2}{2} \frac{\partial}{\partial T} \left( \int_0^T b^2(s, T)ds \right)
\]

\[
= e^{-\kappa T} \int_0^T \theta(s)e^{\kappa s} ds - \frac{\sigma^2}{2} \frac{\partial}{\partial T} \left( \int_0^T b^2(s, T)ds \right). \quad (123)
\]
Rendering Spot Rate Models

Now, using (119), equation (122) may be written

\[
a(0, T) = \int_0^T \frac{1}{\kappa} (1 - e^{-\kappa(T-s)}) \theta(s) ds - \frac{1}{2} \int_0^T \sigma^2 b^2(s, T) ds \quad (124)
\]

\[
= \frac{1}{\kappa} \int_0^T \theta(s) ds - \frac{e^{-\kappa T}}{\kappa} \int_0^T e^{\kappa s} \theta(s) ds - \frac{1}{2} \int_0^T \sigma^2 b^2(s, T) ds.
\]

Using (124) to eliminate the \(e^{-\kappa T} \int_0^T \theta(s) e^{\kappa s} ds\) term in (123) we obtain

\[
\frac{\partial a(0, T)}{\partial T} = -\kappa a(0, T) + \int_0^T \theta(s) ds - \frac{\kappa \sigma^2}{2} \int_0^T b^2(s, T) ds - \frac{\sigma^2}{2} \frac{\partial}{\partial T} \left( \int_0^T b^2(s, T) ds \right),
\]

which upon re-arrangement yields

\[
\int_0^T \theta(s) ds = \frac{\partial a(0, T)}{\partial T} + \kappa a(0, T) + \frac{\sigma^2}{2} \left[ \kappa \int_0^T b^2(s, T) ds + \frac{\partial}{\partial T} \left( \int_0^T b^2(s, T) ds \right) \right].
\]
Rendering Spot Rate Models

Differentiating the last equation with regard to $T$ yields,

$$\theta(T) = \frac{\partial}{\partial T} \left[ \frac{\partial a(0, T)}{\partial T} + \kappa a(0, T) \right] + \frac{\sigma^2}{2} \frac{\partial}{\partial T} \left[ \kappa \int_0^T b^2(s, T) ds + \frac{\partial}{\partial T} \left( \int_0^T b^2(s, T) ds \right) \right].$$

With the function $\theta(T)$ now at our disposal we can compute the time function $a(t, T)$ and hence bond prices calibrated to market data.

In the case of the CIR model, the steps taken to calibrate the model to the initial yield curve and cap and swaption data for example are similar to the Hull-White model. We do not provide details here.
Appendix 1. Solution of the Ordinary Differential Equation (59)
Solution of the Ordinary Differential Equation

Solving for $b(t, T)$

Consider the ordinary differential eqn.

$$\frac{db}{dt} = \alpha_0 b^2 + \alpha_1 b - 1 = \alpha_0 \left[ b^2 + \frac{\alpha_1}{\alpha_0} b - \frac{1}{\alpha_0} \right]. \quad (125)$$

The quadratic in the brackets on the RHS can be factorised as

$$b^2 + \frac{\alpha_1}{\alpha_0} b - \frac{1}{\alpha_0} = (b - \phi_1)(b - \phi_2),$$
Solution of the Ordinary Differential Equation

where

\[ \phi_1 = -\frac{\alpha_1}{2\alpha_0} + \frac{\beta}{2\alpha_0}, \]
\[ \phi_2 = -\frac{\alpha_1}{2\alpha_0} - \frac{\beta}{2\alpha_0}, \]
\[ \beta = \sqrt{\alpha_1^2 + 4\alpha_0}. \]  \hspace{3cm} (126)

Thus the ordinary differential eqn. (125) can be written

\[ \frac{db}{dt} = \alpha_0(b - \phi_1)(b - \phi_2), \]  \hspace{3cm} (127)

\[ \sim \text{ or as} \]

\[ \frac{db}{(b - \phi_1)(b - \phi_2)} = \alpha_0 dt. \]
Solution of the Ordinary Differential Equation

- With some slight re-arrangement the last eqn. can be written

\[
\left[ \frac{1}{b - \phi_1} - \frac{1}{b - \phi_2} \right] db = \alpha_0 (\phi_1 - \phi_2) dt = \beta dt. \quad (128)
\]

- Integrating the last eqn. from \( t \) to \( T \) we obtain

\[
\left[ \ln \left( \frac{b - \phi_1}{b - \phi_2} \right) \right]^{T}_t = \beta (T - t), \quad (129)
\]

i.e.

\[
\ln \left( \frac{b(T, T) - \phi_1}{b(T, T) - \phi_2} \right) - \ln \left( \frac{b(t, T) - \phi_1}{b(t, T) - \phi_2} \right) = \beta (T - t),
\]

\sim \text{ which on making use of } b(T, T) = 0 \text{ becomes }

\[
\ln \left( \frac{b(t, T) - \phi_1}{b(t, T) - \phi_2} \right) = \ln \left( \frac{\phi_1}{\phi_2} \right) - \beta (T - t),
\]
Solution of the Ordinary Differential Equation

\[ \frac{b(t, T) - \phi_1}{b(t, T) - \phi_2} = \exp \left[ \ln \left( \frac{\phi_1}{\phi_2} \right) - \beta(T - t) \right] \]

\[ = \frac{\phi_1}{\phi_2} e^{-\beta(T-t)}. \]

Solving the last eqn. for \( b(t, T) \) we obtain

\[ b(t, T) = \frac{\phi_1 \phi_2 (1 - e^{-\beta(T-t)})}{\phi_2 - \phi_1 e^{-\beta(T-t)}}. \]  \hspace{1cm} (130)

Using the fact that \( \phi_1 \phi_2 = -1/\alpha_0 \) this simplifies slightly to

\[ b(t, T) = \frac{1}{\alpha_0} \frac{[1 - e^{-\beta(T-t)}]}{[\phi_1 e^{-\beta(T-t)} - \phi_2]}. \]  \hspace{1cm} (131)
Solution of the Ordinary Differential Equation

Solving for \( a(t, T) \)

- From equation (62) of Section ??

\[
a(t, T) = \alpha \gamma \int_t^T b(s, T) ds.
\]

- Making the transformation \( u = \beta (T - s) \) we see that

\[
a(t, T) = \frac{+\alpha \gamma}{\beta} \int_0^\beta (T-t) b(T - \frac{u}{\beta}, T) du.
\]

- Substituting the expression (131) for \( b(t, T) \) (and setting \( \tau = T - t \)) we obtain

\[
a(t, T) = + \frac{\alpha \gamma}{\beta} \frac{2}{\sigma^2} \int_0^{\beta \tau} \frac{(1 - e^{-u})}{(\phi_1 e^{-u} - \phi_2)} du.
\]
Solution of the Ordinary Differential Equation

Consider the integral

\[ I = \int_{0}^{\beta \tau} \left( \frac{1 - e^{-u}}{\phi_1 e^{-u} - \phi_2} \right) du \]

\[ = \int_{0}^{\beta \tau} \frac{du}{\phi_1 e^{-u} - \phi_2} - \int_{0}^{\beta \tau} \frac{e^{-u} du}{\phi_1 e^{-u} - \phi_2} \]

\[ = \left[ \frac{1}{\phi_2} \ln(\phi_1 - \phi_2 e^u) \right]_{0}^{\beta \tau} + \left[ \frac{1}{\phi_1} \ln(\phi_1 e^{-u} - \phi_2) \right]_{0}^{\beta \tau} \]

\[ = \frac{(\phi_1 - \phi_2)}{\phi_1 \phi_2} \ln(\phi_1 - \phi_2) - \frac{1}{\phi_2} \ln(\phi_1 - \phi_2 e^{\beta \tau}) + \frac{1}{\phi_1} \ln(\phi_1 e^{-\beta \tau} - \phi_2) \]

\[ = \frac{(\phi_1 - \phi_2)}{\phi_1 \phi_2} \ln(\phi_1 - \phi_2) - \frac{1}{\phi_2} \ln(\phi_1 - \phi_2 e^{\beta \tau}) \]

\[ + \frac{1}{\phi_1} \ln[e^{-\beta \tau} / (\phi_1 - \phi_2 e^{\beta \tau})] \]

\[ = -\frac{\beta \tau}{\phi_1} + \frac{(\phi_1 - \phi_2)}{\phi_1 \phi_2} \left\{ \ln(\phi_1 - \phi_2) - \ln(\phi_1 - \phi_2 e^{\beta \tau}) \right\}. \]
Solution of the Ordinary Differential Equation

Finally

\[ I = -\frac{\beta}{\phi_1} \tau - \left( \frac{\phi_1 - \phi_2}{\phi_1 \phi_2} \right) \ln \left( \frac{\phi_1 - \phi_2 e^{\beta \tau}}{\phi_1 - \phi_2} \right), \]

and so

\[ a(t, T) = \frac{2\alpha \gamma}{\beta \sigma^2} \left[ -\beta \frac{(T-t)}{\phi_1} - \left( \frac{\phi_1 - \phi_2}{\phi_1 \phi_2} \right) \ln \left( \frac{\phi_1 - \phi_2 e^{\beta (T-t)}}{\phi_1 - \phi_2} \right) \right]. \]

Using the fact that \( \phi_1 \phi_2 = -2/\sigma^2 \) and \( \phi_1 - \phi_2 = 2\beta/\sigma^2 \) we finally obtain

\[ a(t, T) = \frac{2\alpha \gamma}{\sigma^2} \left[ -\frac{(T-t)}{\phi_1} + \ln \left( \frac{\phi_1 - \phi_2 e^{\beta (T-t)}}{\phi_1 - \phi_2} \right) \right]. \]  (132)
Solution of the Ordinary Differential Equation

Allowing For Time Varying Coefficients

- The steps leading to (128) remain the same as in the constant coefficients case, only now $\phi_1$, $\phi_2$, $\sigma$ and $\beta$ become functions of time. Thus in order to use the same functional form for the solution we need to define

$$\bar{\beta}(t, T) = \frac{1}{T - t} \int_t^T \beta(s) ds.$$ 

- Thus integration of (128) will yield (131) with $\beta$ replaced by $\bar{\beta}(t, T)$. 
Appendix 1. Calculating $\theta(T)$ in the Calibration of the H-W Model
Calculating $\theta(T)$ in the Calibration of the H-W Mode

- From (110) we note that

$$\frac{\partial b}{\partial T} = e^{K(t)-K(T)}.$$

- Differentiating (117) with respect to $T$ yields

$$\frac{\partial}{\partial T} a(0, T) = b(T, T)\theta(T) + \int_0^T e^{K(s)-K(T)}\theta(s)ds$$

$$- \frac{\partial}{\partial T} \left( \frac{1}{2} \int_0^T \sigma^2(s)b(s, T)^2 ds \right)$$

$$= e^{-K(T)} \int_0^T e^{K(s)}\theta(s)ds - \frac{\partial}{\partial T} \left( \frac{1}{2} \int_0^T \sigma^2(s)b(s, T)^2 ds \right)$$
Calculating $\theta(T)$ in the Calibration of the H-W Mode

- Rearranging

$$e^{\mathcal{K}(T)} \frac{\partial}{\partial T} a(0, T) = \int_{0}^{T} e^{\mathcal{K}(s)} \theta(s) ds - e^{\mathcal{K}(T)} \frac{\partial}{\partial T}$$

$$\left( \frac{1}{2} \int_{0}^{T} \sigma^2(s) b(s, T)^2 ds \right).$$

- Differentiating again with respect to $T$ we obtain

$$\frac{\partial}{\partial T} \left( e^{\mathcal{K}(T)} \frac{\partial}{\partial T} a(0, T) \right) = e^{\mathcal{K}(T)} \theta(T)$$

$$\left( \frac{1}{2} \int_{0}^{T} \sigma^2(s) b(s, T)^2 ds \right).$$
Calculating $\theta(T)$ in the Calibration of the H-W Mode

Rearranging this last equation we obtain

$$
\theta(T) = e^{-\kappa(T)} \frac{\partial}{\partial T} \left( e^{\kappa(T)} \frac{\partial}{\partial T} a(0, T) \right) 
+ e^{-\kappa(T)} \frac{\partial}{\partial T} \left( e^{\kappa(T)} \frac{\partial}{\partial T} \left( \frac{1}{2} \int_0^T \sigma^2(s) b(s, T)^2 ds \right) \right).
$$

